

EXISTENCE OF MAXIMAL SURFACE CONTAINING GIVEN CURVE AND SPECIAL SINGULARITY

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ABSTRACT. We give a different formulation for describing maximal surfaces in Lorentz-Minkowski space, \mathbb{L}^3 , using the identification of \mathbb{L}^3 with $\mathbb{C} \times \mathbb{R}$. Further we give a different proof for the singular Björling problem for the case of closed real analytic null curve. As an application, we show the existence of maximal surface which contains a given curve and has a special singularity.

1. INTRODUCTION

Generalised maximal surfaces in Lorentz-Minkowski space \mathbb{L}^3 are spacelike immersions with zero mean curvature and singularities. In this article, we ask: does there exists a generalised maximal surface containing given curve and having a special singularity. We start with the following example.

Let $\alpha(\theta) = (-\frac{3}{4}\cos\theta, -\frac{3}{4}\sin\theta, \ln\frac{1}{2})$ be a spacelike closed real analytic curve. This curve lies on elliptic catenoid, a maximal surface, given by map $F(x, y) = (\frac{x(x^2+y^2-1)}{2(x^2+y^2)}, \frac{y(x^2+y^2-1)}{2(x^2+y^2)}, \ln\sqrt{x^2+y^2})$. We see that

- (1) the map F is defined for all $z = x + iy \neq 0$ and has conelike singularity on $|z| = 1$.
- (2) there is a positive real r_0 , namely $r_0 = \frac{1}{2}$ such that $F(|z| = r_0) = \gamma(\frac{1}{2}e^{i\theta}) := \alpha(\theta)$.

On the other hand if we take $\beta(\theta) = (e^{i\theta}, 1)$, which is a spacelike closed curve, we will see (in section 4) that there does not exist any maximal surface F (parametrised by single chart F defined for all $z \neq 0$) and any $r_0 \neq 1$ such that $F(r_0e^{i\theta}) = \tilde{\beta}(r_0e^{i\theta}) := \beta(\theta)$ and has singularity (not necessarily cone like) at $|z| = 1$.

In this article, we will see that if given curve γ satisfies some conditions then there exists a generalised maximal surface which has property (1) and (2) as above.

This follows from the solution of the Björling problem for maximal surface. Alías, Chaves, and Mira in [2] solved the Björling problem for maximal surfaces. Kim and Yang in [4] introduced the singular Björling problem and proved it for the case of real analytic null curve defined on an open interval. Immediate extension of the

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singular Björling problem and solution for the case of closed curve was discussed by the same authors in [4]. We have revisited this problem for the case of closed analytic curve and given a different proof for this. We believe the technique used in our proof helps to know more about the generalised maximal surface. In particular it helps to solve the problem discussed in the beginning of the introduction.

When one curve is constant and other curve is nonconstant real analytic and closed, we can ask: can we interpolate both curves by some generalised maximal surface such that the point (corresponding to the constant curve) is a singularity. We answer this question under certain conditions on nonconstant real analytic curve. In this article we did not solve interpolation problem in general, namely the question of existence of a maximal surface interpolating two real analytic curves. This we save for future work.

Article is arranged as follows: In section 2, we discussed the maximal surface and its properties in a modified way. In section 3, we revisit the singular Björling problem discussed by Kim and Yang [4] in a different way. In section 4, we discussed the particular case of the interpolation problem.

Our article is self contained and the content of the article is motivated by (in particular for maximal surface) [6],[7],[9],[4],[3] and for minimal surface by [8].

2. MAXIMAL SURFACE

Vector space \mathbb{R}^3 with the metric $dx^2 + dy^2 - dt^2$, denoted by \mathbb{L}^3 , is known as Lorentz-Minkowski space. We identify the vector space structure of \mathbb{L}^3 with $\mathbb{C} \times \mathbb{R}$, by $(x, y, t) \rightarrow (x + iy, t)$ then the metric is $(dx + idy)(dx - idy) - dt^2$.

Definition 2.1. Let $\Omega \subset \mathbb{C}$ be a domain and $F = (u, v, w) : \Omega \rightarrow \mathbb{L}^3$ be a nonconstant, smooth harmonic map such that the coordinate functions u, v, w satisfy the conformality relations (with $z = x + iy$),

$$(2.1) \quad \begin{aligned} u_x^2 + v_x^2 - w_x^2 &= u_y^2 + v_y^2 - w_y^2 \\ u_x u_y + v_x v_y - w_x w_y &= 0 \end{aligned}$$

and on Ω , $|\frac{\partial u}{\partial z}|^2 + |\frac{\partial v}{\partial z}|^2 - |\frac{\partial w}{\partial z}|^2$ does not vanish identically. F is said to be a Generalised maximal surface.

Let $F = (h := u + iv, w)$, where h is the complex coordinate of F , the conformality relations (2.1) is equivalent to

$$h_z \overline{h_z} - w_z^2 = 0.$$

On Ω , nonvanishing of $|\frac{\partial u}{\partial z}|^2 + |\frac{\partial v}{\partial z}|^2 - |\frac{\partial w}{\partial z}|^2$ is equivalent to $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. In view of the above complex representation, we have an equivalent definition of the generalised maximal surface.

Definition 2.2. Let $F = (h, w) : \Omega \rightarrow \mathbb{C} \times \mathbb{R}$ be a smooth map such that $h_{z\bar{z}} = 0$ and $w_{z\bar{z}} = 0$ with $h_z \overline{h_z} - w_z^2 = 0$ and $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. Generalised

maximal surface is the equivalence class of map F , where equivalence relation is change of conformal parameter.

Example 2.3 (Elliptic catenoid). Let $\Omega = \mathbb{C} - \{0\}$ and $h(z) = \frac{1}{2} \left(z - \frac{1}{z} \right)$, $w(z) = \frac{1}{2} \log(z\bar{z})$. Then we define $F : \mathbb{C} - \{0\} \rightarrow \mathbb{C} \times \mathbb{R}$, $F(z) = (h(z), w(z))$. We have $h_z \overline{h_z} - w_z^2 = 0$ and $h_{z\bar{z}} = w_{z\bar{z}} = 0$ for all $z \in \Omega$. Here $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. Only on $|z| = 1$, $|h_z| = |h_{\bar{z}}|$.

Example 2.4. If we take $h : \mathbb{C} \rightarrow \mathbb{C}$ defined by $h(z) = \sin z + \sin \bar{z} + i0$ and $w(z) = \sin z + \sin \bar{z}$, then we have $h_z \overline{h_z} - w_z^2 = 0$ and $h_{z\bar{z}} = w_{z\bar{z}} = 0$ for all $z \in \mathbb{C}$, but $|h_z|$ is identically equal to $|h_{\bar{z}}|$ on whole of \mathbb{C} . Therefore it is not a generalised maximal surface.

Since F is a generalised maximal surface in isothermal parameters, we have $\langle F_x, F_x \rangle = \langle F_y, F_y \rangle = \eta$, $\langle F_x, F_y \rangle = 0$, we have

$$(2.2) \quad ds^2 = \eta(z)(dx^2 + dy^2) = \eta(z)|dz|^2, \quad \text{where}$$

$$(2.3) \quad \begin{aligned} \eta(z) &= \langle F_x, F_x \rangle \\ &= \langle (h_z + h_{\bar{z}}, w_z + w_{\bar{z}}), (h_z + h_{\bar{z}}, w_z + w_{\bar{z}}) \rangle \\ &= (h_z + h_{\bar{z}})(\overline{h_z + h_{\bar{z}}}) - (w_z + w_{\bar{z}})^2 \end{aligned}$$

Now using conformality relation $h_z \overline{h_z} - w_z^2 = 0$, we obtain

$$\eta(z) = (|h_z| - |h_{\bar{z}}|)^2.$$

A point of $\Omega \subseteq \mathbb{C}$ on which the equation $|h_z| = |h_{\bar{z}}|$ holds is called a singular point of (F, Ω) and set of all *singular points* is called the *singular set* of the maximal surface (F, Ω) . Based on image of singularity set, authors in [4], [9], [5], [12] have discussed various kind of singularities, such as shrinking, curvilinear singularity, cuspidal edges, swallowtails etc. Fernández, López, and Souam in [7] discussed two type of isolated singularity; branch and special singularity. We also use the name special singularity for the singularity as defined below.

Definition 2.5 (Special singularity). *A point p in \mathbb{L}^3 is such that $F(\{|z| = r\}) = p$ for some $r > 0$, then we say that at p the generalised maximal surface (F, Ω) has special singularity, if $|z| = r$ is a subset of the singular set (set of all singular points) of (F, Ω) .*

If (F, Ω) has a special singularity at a point p for $|z| = r$, we often refer to it as p or $|z| = r$.

Points where $|h_z| \neq |h_{\bar{z}}|$ holds are called regular point of (F, Ω) in the sense that at those points of Ω , F will be immersion. We have following easy observation that if F is not an immersion, then in particular $u_x v_y - u_y v_x = 0$. In turn $u_x v_y - u_y v_x = |h_z|^2 - |h_{\bar{z}}|^2$. Thus $|h_z| = |h_{\bar{z}}|$.

Conversely, suppose $|h_z| = |h_{\bar{z}}|$, as $F = (h, w)$ is a generalized maximal surface then $|h_z| = |h_{\bar{z}}|$ corresponds to singular set of the surface. Indeed, since we have $h_z \bar{h}_{\bar{z}} - w_z^2 = 0$, this imply $|w_z|^2 = |h_z|^2 = |h_{\bar{z}}|^2$. This also gives

$$(2.4) \quad 2(u_x v_y - u_y v_x) = (u_x^2 + v_x^2 - w_x^2) + (u_y^2 + v_y^2 - w_y^2)$$

As we have F maximal, so by definition F is spacelike. Therefore, the vectors $F_x = (u_x, v_x, w_x)$ and $F_y = (u_y, v_y, w_y)$ are spacelike vectors and hence

$$\begin{aligned} |F_x|^2 &= u_x^2 + v_x^2 - w_x^2 \geq 0 \\ |F_y|^2 &= u_y^2 + v_y^2 - w_y^2 \geq 0. \end{aligned}$$

therefore we get $|F_x|^2 + |F_y|^2 = 0$. This imply $F_x = F_y = 0$. Thus F is not an immersion. Therefore we see that $F = (h, w) : \Omega \rightarrow \mathbb{L}^3$ is a generalised maximal surface. F is immersion at $p \in \Omega$ if and only if at p , $|h_z| \neq |h_{\bar{z}}|$.

With this representation of maximal surface, following [8], we have the following:

Proposition 2.1. *Let $h : \Omega \rightarrow \mathbb{C}$ be the complex coordinate of the isothermal representation of a generalized maximal surface $F = (h, w) : \Omega \rightarrow \mathbb{C} \times \mathbb{R} \simeq \mathbb{L}^3$. Then on $\Omega \subset \mathbb{C}$, we can write*

$$w(z) = 2\operatorname{Re} \int_{z_0}^z \sqrt{h_z \bar{h}_{\bar{z}}} dz + w(z_0),$$

where the line integral is along any smooth curve starting from z_0 and ending at z .

Proof. The function $h_z \bar{h}_{\bar{z}}$ admits a continuous branch of square root in Ω . Let Γ be a closed curve in Ω . Consider

$$2\operatorname{Re} \int_{\Gamma} \sqrt{h_z \bar{h}_{\bar{z}}} dz = \int_{\Gamma} \sqrt{h_z \bar{h}_{\bar{z}}} dz + \int_{\Gamma} \overline{\sqrt{h_z \bar{h}_{\bar{z}}}} dz = \int_{\Gamma} \omega_z dz + \int_{\Gamma} \bar{\omega}_{\bar{z}} d\bar{z} = \int_{\Gamma} dw = 0.$$

Therefore we have for every closed curve $\Gamma \subset \Omega$,

$$\operatorname{Re} \int_{\Gamma} \sqrt{h_z \bar{h}_{\bar{z}}} dz = 0.$$

This allows us to take $w(z) - w(z_0) = 2\operatorname{Re} \int_{z_0}^z \sqrt{h_z \bar{h}_{\bar{z}}} dz$. This gives

$$w(z) = 2\operatorname{Re} \int_{z_0}^z \sqrt{h_z \bar{h}_{\bar{z}}} dz + w(z_0).$$

□

The complex coordinate representation (as in definition (2.2) and in proposition (2.1)) of the generalised maximal surface helps us to construct various examples of maximal surfaces. In particular if we take any complex harmonic map $h : \Omega \rightarrow \mathbb{C}$ such that $|h_z|$ is not identically same as $|h_{\bar{z}}|$, then the map defined $F : \Omega \rightarrow \mathbb{L}^3$, defined by $F(z) = \left(h(z), 2\operatorname{Re} \int_{z_0}^z \sqrt{h_z \bar{h}_{\bar{z}}} dz \right)$ is a generalised maximal surface.

Example 2.6. If we take $h : \mathbb{C} \rightarrow \mathbb{C}$ defined by $h(z) = e^z + \bar{z}$. Then $h_z = e^z, h_{\bar{z}} = 1$ and hence $|h_z| = |h_{\bar{z}}| = 1$ on imaginary axis. Here $|h_z|$ is not identically equal to $|h_{\bar{z}}|$, by proposition (2.1), we can determine the third real coordinate w to make (h, w) a maximal surface.

$$w(z) = 2\operatorname{Re} \int \sqrt{h_z \overline{h_{\bar{z}}}} dz = 2(e^{\frac{z}{2}} + e^{\frac{\bar{z}}{2}}).$$

The map $F : \mathbb{C} \rightarrow \mathbb{L}^3$ given by $F(z) = (h(z), w(z))$ satisfies $h_z \overline{h_{\bar{z}}} - w_z^2 = 0$ (conformality relations) and $h_{z\bar{z}} = 0, w_{z\bar{z}} = 0$ (harmonicity) and hence defines a generalized maximal surface.

Example 2.7. If we take $h(z) = \frac{1}{2}(z - \frac{1}{\bar{z}})$, by proposition (2.1), we get $w(z) = \frac{1}{2} \log(z\bar{z})$. Then $F(z) = (h(z), w(z))$ defines what is known as a elliptic catenoid which is a generalised maximal surface with singular set the unit circle $\{|z| = 1\}$.

Normal vector at a regular point of a generalised maximal surface can be given by a map $N : \Omega \rightarrow \mathbb{H}^2 := \{(x, y, t) \in \mathbb{L}^3 : x^2 + y^2 - t^2 = -1\}$,

$$(2.5) \quad N(z) = \frac{F_x \times F_y}{|F_x \times F_y|} = \left(\frac{2\sqrt{h_z \overline{h_{\bar{z}}}}}{|h_{\bar{z}}| - |h_z|}, \frac{|h_{\bar{z}}| + |h_z|}{|h_{\bar{z}}| - |h_z|} \right).$$

For a generalised maximal surface (F, Ω) , Ω has three parts $\mathcal{A} := \{z : |h_{\bar{z}}| < |h_z|\}$, $\mathcal{B} := \{z : |h_{\bar{z}}| = |h_z|\}$ and $\mathcal{C} := \{z : |h_{\bar{z}}| > |h_z|\}$. As we defined earlier, \mathcal{B} denotes the singular set of (F, Ω) . The Gauss map at regular points (that is on \mathcal{A} and on \mathcal{C}) is obtained by stereographic projection of N as in (2.5) from \mathbb{H}^2 to \mathbb{C} . It is given by

$$(1) \quad \nu(z) = \sqrt{\frac{h_z}{h_{\bar{z}}}} \text{ on } \mathcal{A}$$

$$(2) \quad \nu(z) = -\sqrt{\frac{h_{\bar{z}}}{h_z}} \text{ on } \mathcal{C}.$$

3. SINGULAR BJÖRLING PROBLEM

Let

$$(3.1) \quad \gamma(e^{i\theta}) = ((\gamma_1 + i\gamma_2)(e^{i\theta}), \gamma_3(e^{i\theta}))$$

$$L(e^{i\theta}) = ((L_1 + iL_2)(e^{i\theta}), L_3(e^{i\theta}))$$

be such that $\langle \gamma', L \rangle = 0$, where γ is a null real analytic closed curve and L is a null real analytic vector field and that atleast one of γ' and L is not identically zero, γ and L both are defined over S^1 . The above data is known as singular Björling data for closed curve. Kim and Yang in [4] studied the singular Björling problem in detail. In this section we will discuss the same problem for the closed curve from a different point of view. The singular Björling problem asks for the existence of a generalised maximal surface

$$F = (h, w) : A(r, R) \rightarrow \mathbb{L}^3$$

such that $F(e^{i\theta}) = \gamma(e^{i\theta})$ and $\frac{\partial F}{\partial \rho} \Big|_{e^{i\theta}} = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta})) = L(e^{i\theta})$ with singular set atleast $\{|z| = 1\}$.

For the existence of maximal surface, having prescribed data as above, we will be looking for complex harmonic maps h and w on some annulus $A(r, R)$, $r < 1 < R$ such that they satisfy

- (1) $h_z \overline{h_z} - w_z^2 \equiv 0$
- (2) $|h_z| = |h_{\bar{z}}|$ on $z = e^{i\theta}$
- (3) $|h_z| - |h_{\bar{z}}|$ is not identically zero on $A(r, R)$.

We have the following relation between first order partial differentials in system (z, \bar{z}) to the first order partial differential in system (ρ, θ) ; where $z = \rho e^{i\theta}$:

$$(3.2) \quad h_z = \frac{1}{2} \left(h_\rho - \frac{i}{\rho} h_\theta \right) e^{-i\theta} \text{ and } h_{\bar{z}} = \frac{1}{2} \left(h_\rho + \frac{i}{\rho} h_\theta \right) e^{i\theta}.$$

Here we have given $(h_\rho, w_\rho) = (L_1 + iL_2, L_3)$ and $(h_\theta, w_\theta) = (\gamma_1' + i\gamma_2', \gamma_3')$ on the unit circle. On $\{|z| = 1\}$, we define the maps g_1 and g_2 as

$$(3.3) \quad g_1(e^{i\theta}) = \sqrt{\frac{L_1 + iL_2}{L_1 - iL_2}}; \text{ if } \gamma' \text{ vanishes identically.}$$

$$(3.4) \quad g_2(e^{i\theta}) = \sqrt{\frac{\gamma_1' + i\gamma_2'}{\gamma_1' - i\gamma_2'}}; \text{ if } L \text{ vanishes identically.}$$

Since γ' and L are dependent (being null vector field and perpendicular), if both γ' and L do not vanish identically, we have $g_1(e^{i\theta}) = g_2(e^{i\theta})$. Therefore we get a well defined map g on S^1 given by g_1 and g_2 as above.

If there exists a generalised maximal surface (h, w) for the given Björling data, then analytic extension of g agrees with $\nu = \sqrt{\frac{h_z}{h_{\bar{z}}}}$ on \mathcal{A} (that is at those points of the domain where $|h_{\bar{z}}| < |h_z|$). Similarly there is a real analytic function on S^1 whose analytic extension matches with ν on \mathcal{C} .

We have the following existence theorem.

Theorem 3.1. *Given a real analytic null closed curve $\gamma : S^1 \rightarrow \mathbb{L}^3$ and a null vector field $L : S^1 \rightarrow \mathbb{L}^3$ such that $\langle \gamma', L \rangle = 0$; atleast one of γ' and L do not vanish identically. If $|g(z)|$ ($g(z)$ is analytic extension of $g(e^{i\theta})$) is not identically equal to 1, then there exists a unique generalised maximal surface $F := (h, w)$ defined on some annulus $A(r, R) := \{z : 0 < r < |z| < R\}; r < 1 < R$, such that*

- (1) $F(e^{i\theta}) = (h(e^{i\theta}), w(e^{i\theta})) = \gamma(e^{i\theta})$.
- (2) $\frac{\partial F}{\partial \rho} \Big|_{e^{i\theta}} = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta})) = L(e^{i\theta})$.

with singular set atleast $\{|z| = 1\}$.

Proof. We will prove this theorem in two steps

- (1) We show the existence of generalised maximal surface containing the given singular Björling data.
- (2) Next we show that the determined generalized maximal surface will have singularity set atleast $\{|z| = 1\}$.

In the step 1, we find a complex harmonic function h and a real harmonic function w defined on some annulus $A(r, R)$, and show that $h_z \overline{h_z} - w_z^2 \equiv 0$. Any harmonic function over some annulus $A(r, R)$ has the following form

$$(3.5) \quad h(z) = \sum_{-\infty}^{\infty} a_n z^n + \frac{b_n}{\bar{z}^n} + c \ln |z|^2.$$

Therefore, in (ρ, θ) coordinates, on the unit circle,

$$(3.6) \quad h_\theta(e^{i\theta}) = i \sum_{-\infty}^{\infty} n(a_n + b_n) e^{in\theta}$$

$$(3.7) \quad h_\rho(e^{i\theta}) = \sum_{-\infty}^{\infty} n(a_n - b_n) e^{in\theta} + c$$

From the given data, $h(e^{i\theta}) = \gamma_1(\theta) + i\gamma_2(\theta)$, we know the left hand side of the equation (3.6) and as γ is analytic, $h_\theta(e^{i\theta})$ is analytic so the series (in equation (3.6)) in the right hand side converges.

Next we equate

$$(3.8) \quad h_\rho(e^{i\theta}) = L_1(\theta) + iL_2(\theta)$$

as above, as $L_1 + iL_2$ is analytic, h_ρ is analytic and hence the series in equation (3.7) converges. We have $n(a_n + b_n)$ as the fourier coefficients of h_θ in equation (3.6) for all n , and those for h_ρ are $n(a_n - b_n)$ in equation (3.7), for all n . Therefore we can solve for a_n , b_n and c uniquely and hence we have determined $h(z)$ such that h is harmonic.

In the same way, the harmonic function $w(z)$ can be determined, because we have given $w(e^{i\theta})$ and $w_\rho(e^{i\theta})$.

Now we will show $h_z \overline{h_z} - w_z^2 = 0$ on unit circle with given data. Indeed,

$$(3.9) \quad \overline{h_z} = \frac{1}{2} \left(\overline{h_\rho} - \frac{i}{\rho} \overline{h_\theta} \right) e^{-i\theta}$$

$$(3.10) \quad w_z = \frac{1}{2} \left(w_\rho - \frac{i}{\rho} w_\theta \right) e^{-i\theta}$$

On unit circle we have

$$\begin{aligned} h_z \overline{h_z} &= \frac{1}{4}(h_\rho - ih_\theta)(\overline{h_\rho} - i\overline{h_\theta})e^{-2i\theta} \\ &= \frac{1}{4}(L_1^2 + L_2^2 - \gamma_1'^2 - \gamma_2'^2 - i(L_1 + iL_2)(\gamma_1' - i\gamma_2') - i(\gamma_1' + i\gamma_2')(L_1 - iL_2))e^{-2i\theta} \end{aligned}$$

As L and γ' are null vector fields we have $L_1^2 + L_2^2 = L_3^2$ and $\gamma_1'^2 + \gamma_2'^2 = \gamma_3'^2$, using these identities in above equation we get

$$(3.11) \quad h_z \overline{h_z} = \frac{1}{4}(L_3^2 - \gamma_3'^2 - 2iL_3\gamma_3')e^{-2i\theta}$$

Next

$$(3.12) \quad w_z^2 = \frac{1}{4}(w_\rho - iw_\theta)^2 e^{-2i\theta} = \frac{1}{4}(L_3^2 - \gamma_3'^2 - 2iL_3\gamma_3')e^{-2i\theta}$$

From equation (3.11) and equation (3.12) we see that $h_z \overline{h_z} - w_z^2 = 0$ on the unit circle. As h and w are harmonic functions on an annulus $A(r, R)$, the function $h_z \overline{h_z} - w_z^2$ is complex analytic on $A(r, R)$ which contains the unit circle and hence $h_z \overline{h_z} - w_z^2 \equiv 0$ on annulus.

As we have given in hypothesis, that the analytic extension $g(z)$ of $g(e^{i\theta})$ is such that $|g(z)|$ is not identically 1, which is, equivalent to saying $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. Hence we have found the unique generalised maximal surface $F := (h, w)$.

Now in this last step, we are going to show that the singular set of the generalised maximal surface $F := (h, w)$ contains atleast $\{|z| = 1\}$.

Since $L_3\gamma_3' = L_1\gamma_1' + L_2\gamma_2'$ and γ' and L are null vector field, we have $L_1\gamma_2' = L_2\gamma_1'$.

By the expression (3.2), on the unit circle we have

$$\begin{aligned} |h_z| &= \frac{1}{2}|h_\rho - ih_\theta| = |L_1 + \gamma_2' + i(L_2 - \gamma_1')| \\ (3.13) \quad |h_z|^2 &= \frac{L_1^2 + L_2^2 + \gamma_1'^2 + \gamma_2'^2}{4} + \frac{L_1\gamma_2' - L_2\gamma_1'}{2} \end{aligned}$$

Similarly,

$$(3.14) \quad |h_{\bar{z}}|^2 = \frac{L_1^2 + L_2^2 + \gamma_1'^2 + \gamma_2'^2}{4} - \frac{L_1\gamma_2' - L_2\gamma_1'}{2}$$

Now subtracting equation (3.14) from (3.13) will give

$$|h_z|^2 - |h_{\bar{z}}|^2 = L_1\gamma_2' - L_2\gamma_1' = 0.$$

Thus $|h_z| = |h_{\bar{z}}|$ on unit circle, this proves that our unique generalised maximal surface $F := (h, w)$ will have singularity set atleast $\{|z| = 1\}$. \square

Example 3.2. If $\gamma(\theta) = (c, c, c)$, a constant curve then for any non vanishing null vector field $L(\theta)$ there exists a generalised maximal surface containing the constant curve as singularity. We give an example illustrating this and the proof of the above theorem. When $L(\theta) = (e^{i\theta}, 1) = (h_\rho, w_\rho)$ and $\gamma'(\theta) = (0 + i0, 0) = (h_\theta, w_\theta)$, we will get a generalised maximal surface known as elliptic catenoid. Recall the expressions (3.6) and (3.7), from these we have

$$0 = i \sum_{-\infty}^{\infty} n(a_n + b_n)e^{in\theta} \text{ and}$$

$$e^{i\theta} = \sum_{-\infty}^{\infty} n(a_n - b_n)e^{in\theta} + c$$

This gives $a_1 - b_1 = 1$ and $a_1 + b_1 = 0$ which imply $a_n = 0, b_n = 0, \forall n \neq 1$ and $c = 0$ and hence from the formula (3.5) we get $h(z) = \frac{1}{2} \left(z - \frac{1}{\bar{z}} \right)$. To obtain $w(z)$, we repeat the same step as in the case of obtaining $h(z)$, because here we know $w_\rho = 1$ and $w_\theta = 0$, from this we get $c = 1$ and $a_n = b_n = 0, \forall n$. This gives $w(z) = \frac{1}{2} \log(z\bar{z})$. Expressions (h, w) together represents an elliptic catenoid.

4. EXISTENCE OF MAXIMAL SURFACE CONTAINING A PRESCRIBED CURVE AND SPECIAL SINGULARITY

We start with an example to explain the problem and possible solution. Let $\tilde{\gamma}(\theta) = (c_1 e^{i\theta}, c_2)$, c_1 and c_2 be some constants. We will see that if we take $c_1 = c_2 = 1$, i.e. $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, then there does not exist any positive real $r_0 \neq 1$ and generalised maximal surface F as in the Definition (2.2), defined on some annulus having $|z| = 1$ such that $F(r_0 e^{i\theta}) = (e^{i\theta}, 1)$ and F restricted to unit circle has a special singularity.

While, in particular, if we take $c_1 = -\frac{3}{4}$ and $c_2 = \ln \frac{1}{2}$, then for $r_0 = \frac{1}{2}$, there is a generalised maximal surface $F : \mathbb{C} - \{0\} \rightarrow \mathbb{L}^3$ such that $F(r_0 e^{i\theta}) = \gamma(r_0 e^{i\theta}) := \tilde{\gamma}(\theta)$, maximal surface is the elliptic catenoid discussed in the example (2.3).

Now below we will verify the above facts in detail. We are looking for the generalised maximal surface F such that

$$(4.1) \quad F(r_0 e^{i\theta}) = (c_1 e^{i\theta}, c_2); r_0 \neq 1, c_1, c_2 \text{ are constants and } F(e^{i\theta}) = (0, 0, 0).$$

Also on $|z| = 1$, F admits singularity.

Suppose if we can find such a maximal surface $F(z) = (h(z), w(z))$ which satisfy the initial data given in (4.1), then h for the maximal surface over an annulus is of the form as in equation (3.5), and similarly for w . The initial condition $F(e^{i\theta}) = (0 + 0i, 0)$ will give us

$$(4.2) \quad a_n + b_n = 0; \forall n$$

and the condition $F(r_0 e^{i\theta}) = (c_1 e^{i\theta}, c_2)$

$$(4.3) \quad a_n r_0^n + \frac{b_n}{r_0} = 0 \Rightarrow a_n = b_n = 0; \forall n \neq 1, 0.$$

$$(4.4) \quad a_0 + b_0 + c \log r_0 = 0 \Rightarrow c = 0 \text{ if } r_0 \neq 1.$$

We use (4.2), (4.3) and (4.4) to get

$$(4.5) \quad a_1 = \frac{r_0}{r_0^2 - 1} c_1 \text{ and } b_1 = -a_1,$$

hence

$$(4.6) \quad h(z) = \frac{r_0 c_1}{r_0^2 - 1} \left(z - \frac{1}{\bar{z}} \right)$$

Similarly for $w(z)$ using initial conditions (4.1) we get

$$(4.7) \quad c_n + d_n = 0; \forall n$$

$$(4.8) \quad c_n r_0^n + \frac{d_n}{r_0^n} = 0 \Rightarrow c_n = d_n = 0; \forall n \neq 1, 0.$$

$$(4.9) \quad c_0 + d_0 + d \log r_0 = c_2 \Rightarrow d = \frac{c_2}{\log r_0} \text{ if } r_0 \neq 1.$$

$$(4.10) \quad w(z) = \left(\frac{c_2}{2 \log r_0} \right) \log z \bar{z}$$

Now in order to have $F(z) = (h(z), w(z))$ as the generalised maximal surface, h and w have to satisfy the conditions given in Definition (2.2). The relation $h_z \bar{h}_{\bar{z}} - w_z^2 \equiv 0$ gives us a relation between c_1, c_2 and r_0 as follows

$$(4.11) \quad \frac{c_1 r_0}{r_0^2 - 1} = \frac{c_2}{2 \log r_0}$$

and we see for any set of constants c_1, c_2, r_0 , satisfies above relation, $|h_z|$ is not identically same as $|h_{\bar{z}}|$. Therefore if we have constants $(c_1, c_2, r_0 \neq 1)$ such that they satisfies (4.11), then there is a generalised maximal surface satisfying initial data (4.1) and having singularity on $|z| = 1$.

Moreover, we see that, for the spacelike closed curve $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, $c_1 = c_2 = 1$, mentioned in the beginning of this section, the equation (4.11) has no solution for any r_0 . Therefore, there does not exists any generalised maximal surface F such that

$$(4.12) \quad F(r_0 e^{i\theta}) = (e^{i\theta}, 1); r_0 \neq 1, F(e^{i\theta}) = (0, 0, 0)$$

and F restricted to unit circle has a special singularity.

In general we can ask the following: Given a real analytic curve $\tilde{\gamma}(\theta)$, does there exists $F : A(r, R) \rightarrow \mathbb{L}^3$, a generalised maximal surface and $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = \tilde{\gamma}(\theta)$ and F has a special singularity at $|z| = 1$.

For a curve $\tilde{\gamma}(\theta) = \gamma(r_0 e^{i\theta}) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$, $r_0 \neq 1$ (where $f(r_0 e^{i\theta}) \in \mathbb{C}, g(r_0 e^{i\theta}) \in \mathbb{R}$), we define the following modified Fourier coefficients of f and g as

$$(4.13) \quad c = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) d\theta; \quad d = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) d\theta,$$

for $n \neq 0$;

$$(4.14) \quad c_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) e^{-in\theta} d\theta; \quad d_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) e^{-in\theta} d\theta.$$

We see that if $\tilde{\gamma}$ is real analytic (since c_n, d_n, c_{-n} , and d_{-n} all converges to 0), $\limsup |c_{-n}|^{\frac{1}{n}} = 0$, $\limsup |c_n|^{\frac{1}{n}} = 0$, $\limsup |d_{-n}|^{\frac{1}{n}} = 0$ and $\limsup |d_n|^{\frac{1}{n}} = 0$, therefore the following two series converges for all $|z| \neq 0$,

$$(4.15) \quad h(z) = \sum_{-\infty}^{\infty} c_n \left(z^n - \frac{1}{\bar{z}^n} \right) + c \log |z|.$$

$$(4.16) \quad w(z) = \sum_{-\infty}^{\infty} d_n \left(z^n - \frac{1}{\bar{z}^n} \right) + d \log |z|,$$

Now we state the following theorem which is an application to the theorem (3.1).

Theorem 4.1. *Let $\tilde{\gamma}(\theta)$ be a nonconstant closed real analytic spacelike curve. Then there exists $s_0 \neq 1$ and a generalised maximal surface $F : \mathbb{C} - \{0\} \rightarrow \mathbb{L}^3$ such that $F(s_0 e^{i\theta}) := \tilde{\gamma}(\theta)$ and having a special singularity at $(0, 0, 0) \in \mathbb{L}^3$ if and only if there exists $r_0 \neq 1$ and constants c, c'_n, d, d'_n s for the curve $\gamma(r_0 e^{i\theta}) := \tilde{\gamma}(\theta) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$, as in equations (4.13), (4.14) which satisfy the relations:*

$$(4.17) \quad \forall k \neq 0; \quad \sum_{-\infty}^{\infty} 4n(n-k)(c_n \bar{c}_{n-k} - d_n d_{n-k}) + 2k(c_k \bar{c} - c \bar{c}_k - 2d_k d) = 0$$

$$(4.18) \quad \text{and} \quad \sum 4n^2(c_n \bar{c}_n - d_n^2) + c \bar{c} - d^2 = 0.$$

Proof. We start proving only if part. Assume that the constants c, c'_n, d, d'_n s satisfies the conditions (4.17) and (4.18) for the curve $\gamma(r_0 e^{i\theta})$. We claim that h and w given by equation (4.15), (4.16) is the generalised maximal surface satisfying given data. We see that $h(|z| = 1) = 0$; $w(|z| = 1) = 0$ and $\gamma(r_0 e^{i\theta}) = (h(r_0 e^{i\theta}), w(r_0 e^{i\theta})) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$. From equations (4.15) and (4.16), we have

$$h_\rho(e^{i\theta}) = \sum_{-\infty}^{\infty} 2nc_n e^{in\theta} + c; \quad \text{and}$$

$$\begin{aligned}
w_\rho(e^{i\theta}) &= \sum_{-\infty}^{\infty} 2nd_n e^{in\theta} + d \\
h_\rho(e^{i\theta}) \cdot \bar{h}_\rho(e^{i\theta}) &= \left(\sum_{-\infty}^{\infty} 2nc_n e^{in\theta} + c \right) \left(\sum_{-\infty}^{\infty} 2n\bar{c}_n e^{-in\theta} + \bar{c} \right) \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} 4n(n-k)c_n \bar{c}_{n-k} + 2k(c_k \bar{c} - c \bar{c}_k) \right) e^{ik\theta} + c\bar{c}
\end{aligned}$$

Similarly we have

$$w_\rho^2(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} 4n(n-k)d_n d_{n-k} + 4kd_k d \right) e^{ik\theta} + d^2$$

All the series above converges absolutely as f and g are real analytic functions. The series conditions on the constants given in the theorem assures that $h_\rho \bar{h}_\rho - w_\rho^2 = 0$ for $z = e^{i\theta}$, that is to say that (h_ρ, w_ρ) is a null vector field along $|z| = 1$.

By the singular Björling problem for closed curve $\alpha(e^{i\theta}) = (0, 0, 0)$ and $L(e^{i\theta}) = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta}))$ we have a unique maximal surface (h', w') on some $A(r, R)$, $r < 1 < R$ by theorem (3.1). On $A(r, R)$ we have (by uniqueness of (h, w) and (h', w') on $A(r, R)$),

$$h_z \bar{h}_{\bar{z}} - w_z^2 = h'_z \bar{h}'_{\bar{z}} - w'^2_z \equiv 0$$

But this being a complex analytic function on $\mathbb{C} - \{0\}$, $h_z \bar{h}_{\bar{z}} - w_z^2 \equiv 0$. Also as $F(r_0 e^{i\theta})$ is a spacelike curve, this gives that $|h_z|$ is not identically equal to $|h_{\bar{z}}|$, and it proves existence of the required generalised maximal surface. Here we can take $s_0 = r_0$.

Now other way, if the generalized maximal surface F is given such that $F(s_0 e^{i\theta}) = \gamma(\theta)$ and $F(|z| = 1) = (0, 0, 0)$, then F has to be of the form (h, w) as given in the equations (4.15) and (4.16) with $c, c'_n s, d, d'_n s$ as in (4.13) and (4.14) with $r_0 = s_0$. In this form prescribing singularity set as $|z| = 1$ is same as asking for the vector $(h_\rho(e^{i\theta}), w_\rho(e^{i\theta}))$ is a null vector which gives series conditions as in (4.17) and (4.18). \square

For a given spacelike closed curve, $r_0 (\neq 1)$ may not exist as in the above theorem unless it satisfies those series conditions and if such an r_0 exists, it need not be unique. For instance, we have seen that for the curve $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, there does not exists a generalised maximal surface and $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = \tilde{\gamma}(\theta)$; $F(e^{i\theta}) = (0, 0, 0)$ with singular set atleast $|z| = 1$. But if we do some small perturbations of this curve $\tilde{\gamma}$, i.e., for $\epsilon > 0$, let $F(r_0^\epsilon e^{i\theta}) = \tilde{\gamma}_\epsilon(\theta) = ((1 - \epsilon)e^{i\theta}, 1)$, compare this with (4.1), then $\frac{c_1}{c_2} = 1 - \epsilon$, and from the equation (4.11) we see that there are two choice of r_0^ϵ for fixed ϵ . Also, we can see that as $\epsilon \rightarrow 0$, $\tilde{\gamma}_\epsilon \rightarrow \tilde{\gamma}$ and $r_0^\epsilon \rightarrow 1$.

In the above theorem, fixing the special singularity at $(0, 0, 0)$ and asking for the existence of a generalised maximal surface is not necessary, we may ask for any point $(x_1, x_2, x_3) \in \mathbb{L}^3$ as the special singularity corresponding to $|z| = 1$. But then accordingly the expression of h and w as in (4.15) and (4.16) will change and the new series conditions (for e.g. (4.17) and (4.18)) will be found by posing condition that new (h_ρ, w_ρ) is null vector along $|z| = 1$. We believe it is not the statement but the proof of the theorem that gives a handy way to check existence of the generalised maximal surface for a given closed spacelike curve.

Example 4.2. We have seen that for $\tilde{\gamma}(\theta) = (-\frac{3}{4}e^{i\theta}, \ln \frac{1}{2})$, if we take $r_0 = \frac{1}{2}$ then the constants as in equations (4.13), (4.14) are as follows $c_1 = \frac{1}{2}$, $d = 1$ and for all $n \neq 1$, $c'_n s = 0$, $d_n = 0$ and $c = 0$, $d_1 = 0$ and these constants satisfies the series conditions as in equations (4.17) and (4.18). Therefore there exists a generalised maximal surface which is given by expression of h and w as in example (2.3) having special singularity.

Example 4.3. Consider the curve

$$(4.19) \quad \tilde{\gamma}(\theta) = (a_3 e^{3i\theta} + a_1 e^{-i\theta}, b_2 e^{2i\theta} + b_2 e^{-2i\theta}).$$

Below we will analyze for given constants a_1, a_3 , and b_2 , does there exists $r_0 \neq 1$ and the generalised maximal surface as in theorem above ((4.1)).

Recall the formulas (4.13) and (4.14), for $f(r_0 e^{i\theta}) = a_3 e^{3i\theta} + a_1 e^{-i\theta}$ and $g(r_0 e^{i\theta}) = b_2 e^{2i\theta} + b_2 e^{-2i\theta}$, then we have $c_0 = 0$, $c = 0$, for $n \neq -1, 3$; $c_n = 0$ and

$$(4.20) \quad c_{-1} = \frac{r_0^{-1}}{r_0^{-2} - 1} a_1, \quad c_3 = \frac{r_0^3}{r_0^6 - 1} a_3.$$

Similarly $d = 0$, for $n \neq -2, 2$; $d_n = 0$ and

$$(4.21) \quad d_2 = \frac{r_0^2}{r_0^4 - 1} b_2, \quad d_{-2} = \frac{r_0^{-2}}{r_0^{-4} - 1} b_2.$$

Suppose the constants a_1, a_3 and b_2 are such that the curve $\tilde{\gamma}$ is spacelike, then there exists a generalised maximal surface F and r_0 as in theorem (4.1) if and only if the conditions (4.17) and (4.18) are satisfied by the constants $c, c'_n s, d, d'_n s$. That is to say

$$\forall k \neq 0 \quad \sum_{n=-2, -1, 2, 3} 4n(n-k)(c_n c_{n-k} - d_n d_{n-k}) = 0 \quad \text{and} \quad \sum_{n=2, -2, -1, 3} 4n^2(c_n^2 - d_n^2) = 0$$

which is equivalent to

$$(4.22) \quad 4d_2 d_{-2} - 3c_3 c_{-1} = 0 \quad \text{and} \quad c_{-1}^2 + 9c_3^2 = 4(d_2^2 + d_{-2}^2).$$

Therefore for $\gamma(r_0 e^{i\theta}) := \tilde{\gamma}(\theta) = (a_3 e^{3i\theta} + a_1 e^{-i\theta}, b_2 e^{2i\theta} + b_2 e^{-2i\theta})$ if a_1, a_3, b_2 are such that $\tilde{\gamma}$ is spacelike then there exists a maximal surface $F : \mathbb{C} - \{0\} \rightarrow \mathbb{L}^3$ such that

$F(r_0 e^{i\theta}) = \tilde{\gamma}(\theta)$ and F has special singularity at $|z| = 1$ if and only if

$$(4.23) \quad 4 \frac{b_2}{\left(r_0^2 - \frac{1}{r_0^2}\right)} \frac{b_2}{\left(r_0^2 - \frac{1}{r_0^2}\right)} = \frac{3a_3}{\left(r_0^3 - \frac{1}{r_0^3}\right)} \frac{a_1}{\left(r_0 - \frac{1}{r_0}\right)} \text{ and}$$

$$(4.24) \quad \frac{a_1^2}{\left(r_0 - \frac{1}{r_0}\right)^2} + \frac{9a_3^2}{\left(r_0^3 - \frac{1}{r_0^3}\right)^2} = 2 \frac{4b_2^2}{\left(r_0^2 - \frac{1}{r_0^2}\right)^2}.$$

In particular, for any given positive real $c \neq 1$, constants a_1, a_3 and b_2 as $a_1 = \frac{1}{2} \left(c - \frac{1}{c}\right)$, $a_3 = \frac{1}{6} \left(c^3 - \frac{1}{c^3}\right)$ and $b_2 = \frac{1}{4} \left(c^2 - \frac{1}{c^2}\right)$ satisfy the equations (4.23) and (4.24) with $r_0 = c$.

Also for any positive $c \neq 1$, the curve

$$\tilde{\gamma}(\theta) = \left(\frac{1}{2} \left(c - \frac{1}{c}\right) e^{-i\theta} + \frac{1}{6} \left(c^3 - \frac{1}{c^3}\right) e^{3i\theta}, \frac{1}{2} \left(c^2 - \frac{1}{c^2}\right) \cos 2\theta \right)$$

is spacelike. Therefore there is a generalised maximal surface as in theorem (4.1) containing the curve $\tilde{\gamma}$ as above with special singularity at $|z| = 1$. The generalised maximal surface is given by

$$h(z) = \frac{1}{6} \left(z^3 - \frac{1}{\bar{z}^3}\right) + \frac{1}{2} \left(\bar{z} - \frac{1}{z}\right); \quad w(z) = \frac{1}{4} \left(z^2 - \frac{1}{\bar{z}^2} - \frac{1}{z^2} + \bar{z}^2\right).$$

5. CONCLUSION

We have seen how complex representation of generalised maximal surface

$$F(z) = \left(h(z), 2Re \int_{z_0}^z \sqrt{h_z \bar{h}_{\bar{z}}} dz \right),$$

helps us to solve the singular Björling problem (theorem (3.1)) and a particular type of interpolation problem (theorem (4.1)). This representation helps to see that there does not exist any $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = (e^{i\theta}, 1)$ with special singularity on $|z| = 1$, but small perturbation of curve $(e^{i\theta}, 1)$ gives the existence of required r_0 and F .

In general there may exist a maximal surface G defined over a Riemann surface such that G may have special singularity at some point and there may be curve on that Riemann surface whose image is arbitrary curve say for example $\tilde{\gamma} = (e^{i\theta}, 1)$. This is a general problem which still needs to be explored for general curves.

REFERENCES

- [1] J. A. Aledo, J. A. Gálvez, and P. Mira, *Björling Representation for spacelike surfaces with $H = cK$ in \mathbb{L}^3* , Proceedings of the II International Meeting on Lorentzian Geometry, Publ. de la RSME 8 (2004), 2–7.

- [2] L. J. Alías, R. M. B. Chaves, and P. Mira, Björling problem for maximal surfaces in Lorentz-Minkowski space, *Math. Proc. Cambridge Philos. Soc.* 134 (2003), no. 2, 289–316.
- [3] F. J. M. Estudillo and A. Romero, Generalized maximal surfaces in Lorentz-Minkowski space \mathbb{L}^3 , *Math. Proc. Cambridge Philos. Soc.* 111 (1992), no. 3, 515–524.
- [4] Y. W. Kim and S.-D. Yang, Prescribing singularities of maximal surfaces via a singular Björling representation formula, *J. Geom. Phys.* 57 (2007), no. 11, 2167–2177.
- [5] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space \mathbb{L}^3 , *Tokyo J. Math.* 6 (1983), no. 2, 297–309.
- [6] B. ÓNeill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
- [7] I. Fernández, F. López, R. Souam, The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz - Minkowski space \mathbb{L}^3 , *Math. Ann.* 332 (3) (2005) 605–643.
- [8] Tadeusz Iwaniec, Leonid V. Kovalev, and Jani Onninen, Doubly connected minimal surfaces and extremal harmonic mappings, *J. Geom. Anal.* (2012), 22: 726–762.
- [9] S. Fujimori, K. Saji, M. Umehara and K. Yamada, Singularities of maximal surfaces, *Math. Z.*, 259 (2008), 827–848.
- [10] Y. W. Kim, S.-E. Koh, H. Shin and S.-D. Yang, Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae, *Journal of Korean Math. Soc.* 48 (2011), 1083–1100.
- [11] Y. W. Kim and S.-D. Yang, A family of maximal surfaces in Lorentz-Minkowski three-space, *Proc. Amer. Math. Soc.* 134 (2006), 3379–3390.
- [12] M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, *Hokkaido Math. J.*, 35 (2006), 13–40.

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